MATH 579 Exam 4 Solutions

Part I: Recall the difference operator Δ, where $\Delta f(x) = f(x+1) - f(x)$. Define the shift operator E, as $Ef(x) = f(x+1)$. Prove the product rule $\Delta(uv) = u\Delta v + (Ev)\Delta u$. Prove summation by parts: $\sum u\Delta v\delta x = uv - \sum(Ev)\Delta u\delta x$. Find a closed form for $\sum_{k=0}^{n} k2^k$.

$\Delta(u(x)v(x)) = u(x+1)v(x+1) - u(x)v(x) = u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x) = v(x+1)\Delta u + u(x)\Delta v = u\Delta v + (Ev)\Delta u$, which is the product rule. We rearrange to get $u\Delta v = \Delta(uv) - (Ev)\Delta u$, then sum over all x (and use FTDC to get $\sum \Delta(uv) = uv$, with the constant absorbed into one of the other sums) to get the summation by parts formula. Finally, we seek $\sum_0^{n+1} x2^x\delta x$. We seek an anti-difference to $x2^x$. We set $u = x, \Delta v = 2^x$. Hence $\Delta u = 1, v = 2^x, Ev = 2^{x+1}$, so $\sum x2^x\delta x = x2^x - \sum 2^{x+1}1\delta x = x2^x - 2\sum 2^x\delta x = x2^x - 2\cdot 2^x + C = x2^x - 2^{x+1} + C$. In other words, $\Delta(x2^x - 2^{x+1} + C) = x2^x$. By the fundamental theorem of difference calculus, $\sum_{k=0}^{n} k2^k = \sum_0^{n+1} x2^x\delta x = x2^x - 2^{x+1}|_0^{n+1} = (n+1)2^{n+1} - 2^{n+2} - (02^0 - 2^1) = (n+1)2^{n+1} - 2\cdot 2^{n+1} + 2 = (n-1)2^{n+1} + 2$.

Part II:

1. Prove that for $n \in \mathbb{N}_0$, $3^n = \sum_{k=0}^{n} 2^k\binom{n}{k}$.

   This follows directly from Newton's binomial theorem with $x = 2, y = 1$.

2. Prove that for $n \in \mathbb{N}$, $\binom{2n}{n} < 4^n$.

   Here is a combinatorial proof. We choose subsets from $[2n]$. $2^{2n} = 4^n$ counts all possible subsets; $\binom{2n}{n}$ counts only those subsets of size n, which is not all possible subsets (e.g. the empty set is not included).

3. How many northeastern lattice paths are there from $(0,0)$ to $(20,10)$ that do not pass through $(15,5)$?

   The set of northeastern lattice paths from $(0,0)$ to (j,k) is isomorphic with the set of words of length $j+k$ consisting of j N's, and k E's. There are $\binom{j+k}{j}$ such words. Hence there are $\binom{30}{10} = 30,045,015$ paths, ignoring the restriction. We now count how many paths DO pass through $(15,5)$ – they consist of a path from $(0,0)$ to $(15,5)$, followed by a path from $(15,5)$ to $(20,10)$. There are $\binom{20}{5}$ of the former. The latter paths are isomorphic to paths from $(0,0)$ to $(5,5)$, of which there are $\binom{10}{5}$. Hence there are $\binom{20}{5}\binom{10}{5} = 15,504\cdot 252 = 3,907,008$ forbidden paths, and hence $30,045,015 - 3,907,008 = 26,138,007$ desired paths.

4. Prove that for $k,m,n \in \mathbb{N}$, $\binom{n+m}{k} = \sum_{i=0}^{k}\binom{n}{i}\binom{m}{k-i}$.

   Here is a combinatorial proof. We have n numbered red balls, and m numbered blue balls. There are $\binom{n+m}{k}$ ways to choose k balls from the combined set, without regard to color. On the other hand, if we care about color, then let i denote the number of red balls chosen; $k-i$ must be the number of blue balls chosen. As i varies, we get a partition of the selection (sum rule). There are $\binom{n}{i}$ ways to choose the red balls, and $\binom{m}{k-i}$ ways to choose the blue balls. These two selections are independent (product rule).

5. When we expand $(x_1 + x_2 + \cdots + x_m)^n$ fully, what is the largest coefficient?

   Each coefficient will be of the form $\frac{n!}{a_1!a_2!\cdots a_m!}$, where $a_1 + a_2 + \cdots + a_m = n$. We now prove that the maximal coefficient will have $|a_i - a_j| \le 1$, for all i,j. Suppose otherwise, that $a_i \ge a_j + 2$. Well, consider instead $a_i' = a_i - 1, a_j' = a_j + 1$. $\frac{(a_i')!(a_j')!}{(a_i)!(a_j)!} = \frac{a_j+1}{a_i} < 1$ since $a_i \ge a_j + 2$. Hence by replacing a_i, a_j with a_i', a_j' we can make the denominator smaller and the coefficient bigger, contradicting the maximality assumption.
   So, each a_i will equal either $s = \lceil \frac{n}{m} \rceil$, or $t = \lfloor \frac{n}{m} \rfloor$. But how many of each? For this we need the division algorithm: there are q,r such that $n = mq + r$. Hence, the largest coefficient is $\frac{n!}{(s!)^r(t!)^{m-r}} = \frac{n!}{(s!)^m t^r}$.

Exam grades: High score=102, Median score=84, Low score=52